MaxPol Package: Numerical Differentiation for Forward Imaging Problems

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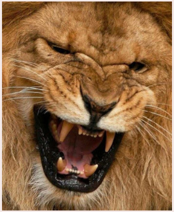
Intuitive Examples from Forward Imaging Applications

Gradients (1st order)

$$\nabla = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix}$$

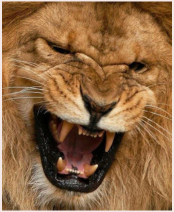
e.g. Image edge strength ($|\nabla I|$)



Hessian (2nd order)

$$H = \begin{bmatrix} \partial^2/\partial x^2 & \partial^2/\partial x \partial y \\ \partial^2/\partial y \partial x & \partial^2/\partial y^2 \end{bmatrix}$$

e.g. Image sharpening ($I - \alpha \nabla^2 I$)



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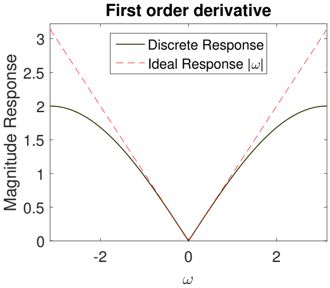
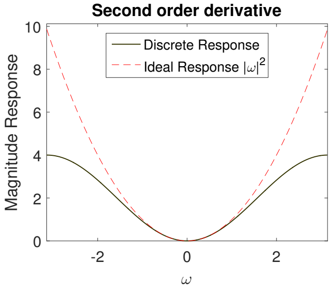
Outline

1. First-order-accuracy is a common practice in image processing

$$\frac{\partial}{\partial x} \approx [-1, 1] \qquad \frac{\partial}{\partial y} \approx \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \frac{\partial^2}{\partial x^2} \approx [1, -2, 1] \qquad \frac{\partial^2}{\partial y^2} \approx \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

2. Corresponding filter responses (DFT)



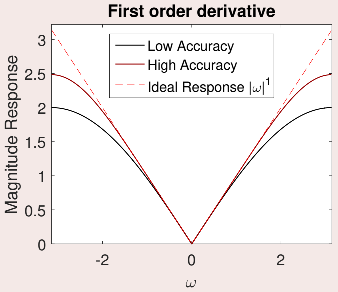


3. Downsides
   - Low approximation accuracy
   - Noise sensitive

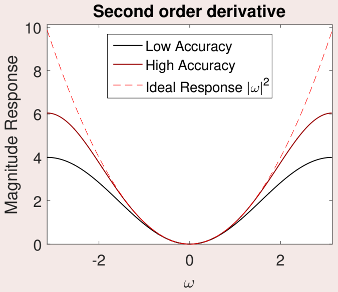
How to increase approximation accuracy?

1. Increase order of polynomials (more tap-lengths)
2. Well known study case in finite difference methods[1]

1st order: $\frac{1}{1920}$ [-9,125,-2250,2250,-125,9]



2nd order: $\frac{1}{180}$ [2,-27,270,-490,270,-27,2]



3. Noise sensitivity increases as a side-effect
4. What should we do then?

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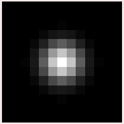
[1] Comprehensive study by [Fornberg 1988, 1998], [LeVeque 2007]

Smooth differentiation in image processing

1. Cascade implementations
   - Apply smoothing kernel (aka lowpass filter) e.g. Gaussian
   - Apply numerical difference kernel e.g. low $\leftrightarrow$ high accuracy

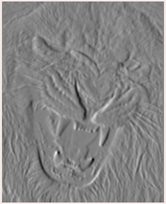
Original Image

Gaussian

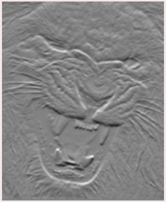


Smooth Image

$\partial I/\partial x$



$\partial I/\partial y$



2. All-in-one implementations
   - 2D Gaussian

     $$-\frac{x}{2\pi\sigma_x^3\sigma_y}\exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right)$$

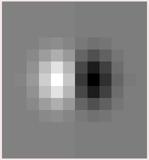
     1. 1st OD Gaussian

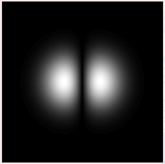
        $$-\frac{x}{\sqrt{2\pi}\sigma_x^3}\exp\left(-\frac{x^2}{2\sigma_x^2}\right)$$

     2. Gaussian $\frac{1}{\sqrt{2\pi}\sigma_y}\exp\left(-\frac{y^2}{2\sigma_y^2}\right)$

   - Sobel, Prewit, and Roberts masks

2D Gaussian

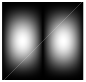




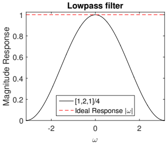
Why Sobel mask is so popular?

1. Recall Sobel 3×3 mask[2] for horizontal differentiation

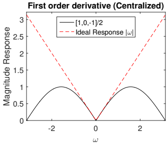
$$\frac{\partial}{\partial x} \approx \frac{1}{8}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix}$$



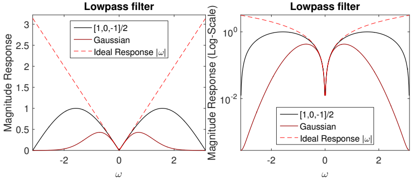
$$\frac{1}{4}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$



$$\frac{1}{2}[1, 0, -1]$$



2. First order centralized derivative is zero on Nyquist frequency
3. Less sensitive to noise as opposed to staggered scheme $[-1, 1]$
4. Order of accuracy can be increased by increasing tap-length
5. Let's compare it with Gaussian derivative ($\sigma = \sqrt{2}$)

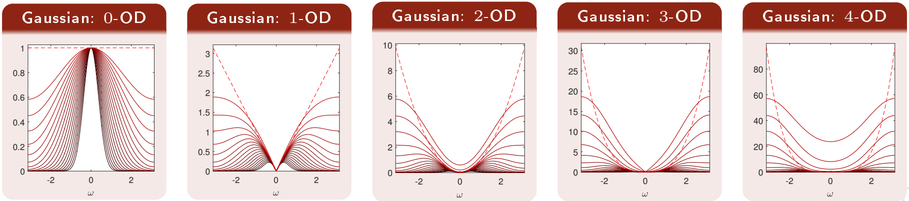


6. Gaussian performs better on canceling noise artifacts

[2] [Sobel 1968]

Gaussian for lowpass differentiation

1. Gaussian kernels can be tweaked in different scales σ
2. Seminal works on scale-space operations[3]
3. Varying σ is equivalent to manipulation of cutoff frequency
4. Remove high frequency components (lowpass filter)

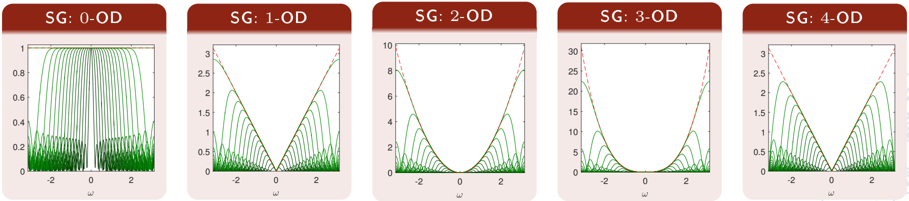


5. Decreasing σ obliterates filter with low-decay response
6. Limited cutoff frequency is offered by Gaussian filters
7. Less accurate on mid-band frequency and higher
8. Many images contain rigid textures

[3] [Lindeberg, 1998, 2011, 2013, 2017], [Lowe 2004]

Design of a proper lowpass differentiation

- Designing requirements
  1. set a costume range of cutoffs $0 \leq \omega_c \leq \pi$
  2. maintain sharp roll-off factor to minimize transition band
  3. observe free-residual artifact on the stop-band
- Sub-optimal solutions via Savitzky-Golay (SG)
  1. polynomial based design in Fourier domain
  2. cutoff is set by truncating high frequency harmonies
  3. minimizes the Gibbs energy in least-square sense

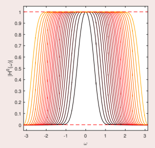


- Side-lobe artifacts are directly tied with human visual perception error

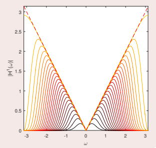
Optimum design by maximally-flat lowpass filters

- Design a linear-phase lowpass FIR filter such that
  1. filter traces the ideal response up to a certain cutoff level
  2. stop-band is flattened by making the response tangent to zero
- Zero and first order designs introduced by Selesnick[4]

Selesnick: 0-OD



Selesnick: 1-OD

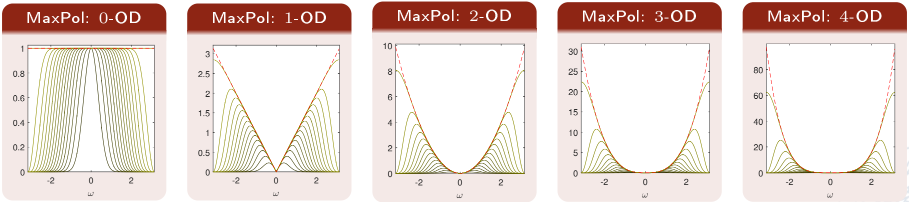


- Refer to `maxflat.m` in MATLAB (aka generalized Butterworth)
- Extension to higher orders requires to solve a non-homogeneous high order nonlinear differential equations
- Many imaging applications require second order differentiation
- E.g. curvature, sharpening, tensor analysis, etc
- Smoothing kernel in Sobel $[1, 2, 1]$ is a MaxFlat lowpass filter!

[4] [Selesnick 1998, 1999, 2002]

Maximally Polynomial (MaxPol) lowpass differentiation

- MaxPol[5] generalizes the design to higher order derivatives
- Incorporates undetermined coefficients in MaxFlat framework
- Closed form solutions accommodated by Taylor series expansion



[5] [Hosseini-Plataniotis 2017]

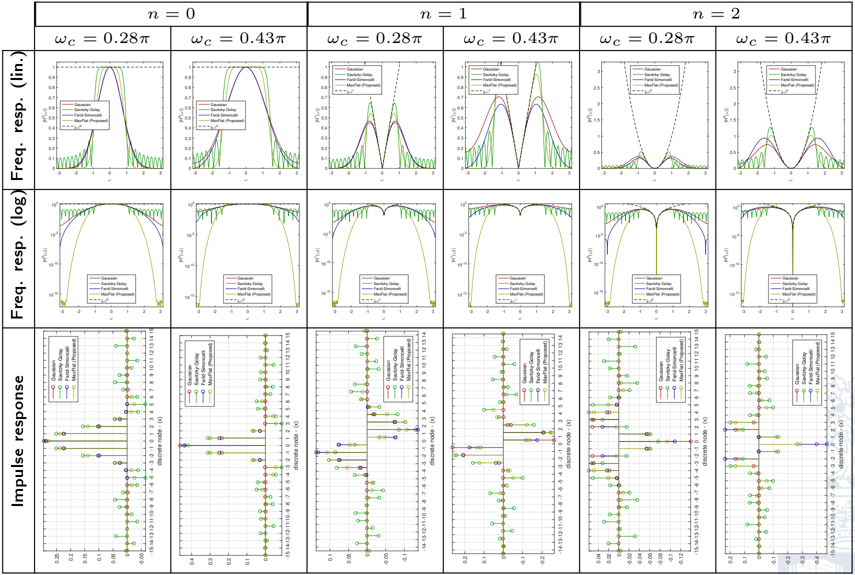
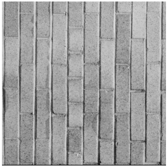
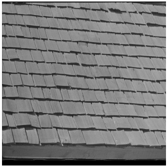
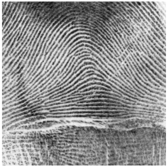

n = 0
n = 1
n = 2
ω_c = 0.28π
ω_c = 0.43π
Freq. resp. (lin.)
Freq. resp. (log)
Impulse response
Gaussian
Savitzky-Golay
Farid-Simoncelli
MaxFlat (Proposed)
discrete node - (x)

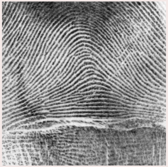
Image directional differentiation

- Textural information in images are oriented
- Consider horizontal, vertical and slanted line features

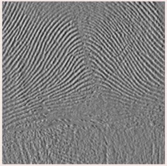




- Maximum edge variation is perpendicular to texture orientation
- Encode by directional differentiation e.g. 1st order $D^1_{\vec{u}} I = \vec{u} \cdot \nabla I$

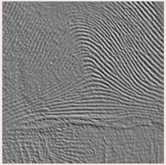
Original Image I



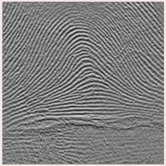
$D^1_0 I$



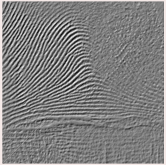
$D^1_{\pi/4} I$



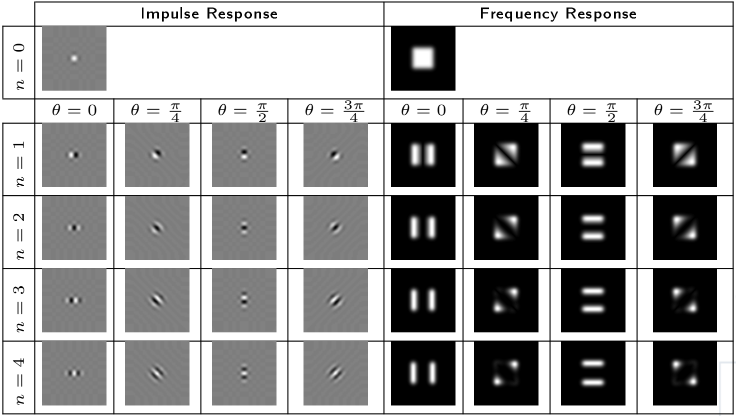
$D^1_{\pi/2} I$



$D^1_{3\pi/4} I$

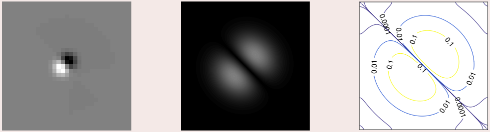


- nth order directional differentiation $D^n_{\vec{u}} I = \sum_{k=0}^{n} \binom{n}{k} u_x^{n-k} u_y^k \frac{\partial^n I}{\partial x^{n-k} \partial y^k}$
- Impulse response obtained by $K_{2D} = \delta(x, y)$
- Substitute MaxPol kernels for nth order differentiation

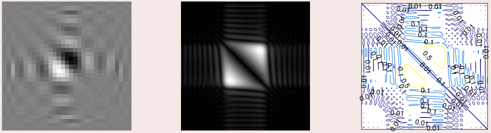


- Compare the sharp localization with steerable Gaussian[6]

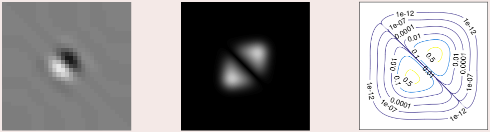
Gaussian, first order differentiation, steering angle $= 45°$, $\omega_c = 1$ (rad)



Savtizky-Golay, first order differentiation, steering angle $= 45°$, $\omega_c = 1$ (rad)



MaxPol, first order differentiation, steering angle $= 45°$, $\omega_c = 1$ (rad)



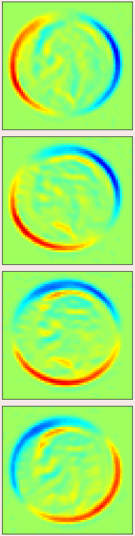
[6] [Freeman-Adelson 1991], [Jacob-Unser 2004]

Example of differentiations along four directions

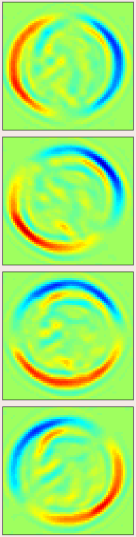
- steering angles
  $\theta = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$

Original Image

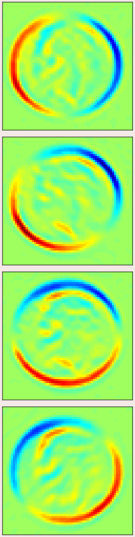
Gaussian



Savtizky-Golay



MaxPol

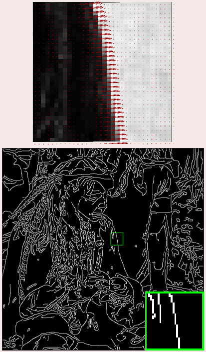


Canny edge detection algorithm

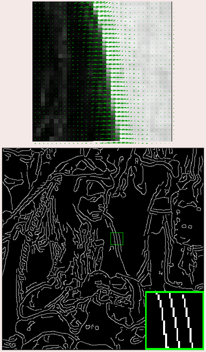
- Canny[7] presents image edges in abstract level
  1. Smooth gradients along horizontal and vertical axes
  2. Edge thinning by non-maximum suppression technique
  3. Edge tracking by hysteresis method
- Use different method of differentiation in (1) ($\omega_c = 0.75\pi$)

Pirate Image

Gaussian



Savitzky-Golay



MaxPol

[7] [Canny 1986]

Unsharp masking

- Enhance image edges by ban-pass filter [8] i.e. $I_{\text{sharp}} = I - \alpha\nabla^2 I$
- Laplacian of Gaussian (LoG) commonly used for sharpening
- Different Gaussian scales represented rigid textures
- Lets substitute MaxPol instead of Gaussian!

Original Image

Gaussian

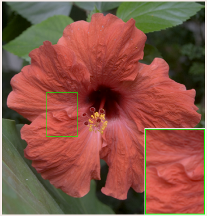
Savitzky-Golay

MaxPol

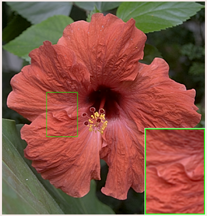
[8] [Rosenfeld-Kak 1982], [Polesel-Ramponi-Mathews 2000]

Image sharpening continued

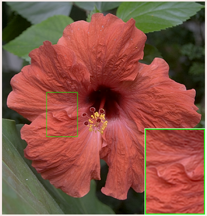
Original



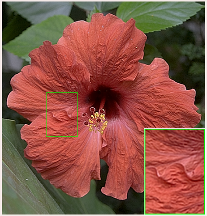
Gaussian



Savitzky-Golay

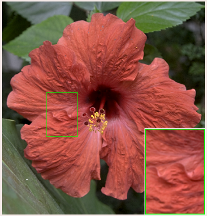


MaxPol

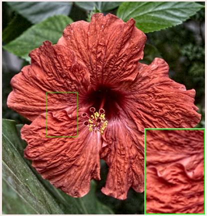


Regular wavelet

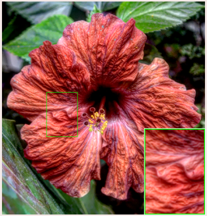
Edge avoid wavelet



Multiscale edge preserving



Mixed domain filtering



Concluding remarks

- MaxPol provides generalized framework for numerical differentiation
  1. arbitrary order of differentiations
  2. flexible cutoff frequency from lowpass to fullband
  3. sharp roll-off
  4. free artifacts on stopband
- MaxPol provides 2D steerable kernels for differentiation
- Powerful toolbox covering different aspects of low-level image processing

Thank You!